Eur. Phys. J. B **37**, 139–142 (2004)
DOI: 10.1140/epjb/e2004-00039-7

THE EUROPEAN PHYSICAL JOURNAL B

Autocorrelation functions of nonlinear Fokker-Planck equations

T.D. Frank[a]

Institute for Theoretical Physics, University of Münster, Wilhelm-Klemm-Str. 9, 48149 Münster, Germany

Received 30 October 2003
Published online 15 March 2004 –

Abstract. We compute autocorrelation functions from nonlinear Fokker-Planck equations that describe nonlinear families of Markov diffusion processes and illustrate this approach for the Plastino-Plastino Fokker-Planck equation related to the Tsallis entropy.

PACS. 05.20.-y Classical statistical mechanics – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

Rapid Note

Nonlinear Fokker-Planck equations have been applied in various fields such as neuro- and biophysics [1–6], human movement sciences [7–9], the theory of reentrant phase transitions, laser arrays, and electronic circuitry, [10–13], the physics of polymer fluids [14], and astrophysics [15,16]. So far, research has primarily been focused on the study of the process distributions defined by nonlinear Fokker-Planck equations while little attention has been paid to the study of correlation functions. Correlation functions, however, contain indispensable information about stochastic processes. In what follows, we will show how to assess this information in general and in the special case of a model proposed by Plastino and Plastino.

We consider systems described by a random variable $X(t) \in \Omega$ defined on a phase space Ω. Let X be distributed like $u(x)$ at an initial time t_0. Then, we assume that the probability density $P(x,t;u) = \langle \delta(x - X(t)) \rangle$ of the systems satisfies for $t = t_0$ the initial condition $P(x,t_0;u) = u(x)$ and for $t \geq t_0$ the nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t;u) = -\frac{\partial}{\partial x}D_1(x,t,P)P + \frac{\partial^2}{\partial x^2}D_2(x,t,P)P\,. \quad (1)$$

We further assume that for solutions of equation (1) the drift and diffusion coefficients D_1 and D_2 can be regarded as first and second Kramers-Moyal coefficients of a (nonlinear) family of Markov diffusion processes [17]. In this case, the initial distribution u is used as a label that describes the family members and the transition probability density $P(x,t|x',t';u)$ of the family of Markov diffusion processes satisfies

$$\frac{\partial}{\partial t}P(x,t|x',t';u) = -\frac{\partial}{\partial x}D_1(x,t,P(x,t;u))P(x,t|x',t';u) + \frac{\partial^2}{\partial x^2}D_2(x,t,P(x,t;u))P(x,t|x',t';u). \quad (2)$$

Joint probability densities can be computed from $P(x,t;u)$ and $P(x,t|x',t';u)$. For example, we have

$$\begin{aligned} &P(x,t;x',t';u) = P(x,t|x',t';u)P(x',t';u),\\ &P(x,t;x',t';x'',t'';u) =\\ &P(x,t|x',t';u)P(x',t'|x'',t'';u)P(x'',t'';u) \end{aligned} \quad (3)$$

for $t \geq t' \geq t''$. Alternatively, the family of Markov diffusion processes is described by the Ito-Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = D_1(x,t,P)|_{x=X(t)} + \sqrt{D_2(x,t,P)}\Big|_{x=X(t)}\,\Gamma(t), \quad (4)$$

where Γ is a Langevin force with $\langle \Gamma \rangle = 0$ and $\langle \Gamma(t)\Gamma(t') \rangle = 2\delta(t-t')$ [18]. For details, see [17]. Next, let us define the autocorrelation functions

$$C^{mn}(t,t') = \langle X^m(t)X^n(t') \rangle\,. \quad (5)$$

Multiplying equation (2) with $P(x',t';u)$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}P(x,t;x',t';u) = &-\frac{\partial}{\partial x}D_1(x,t,P(x,t;u))P(x,t;x',t';u)\\ &+\frac{\partial^2}{\partial x^2}D_2(x,t,P(x,t;u))P(x,t;x',t';u). \end{aligned} \quad (6)$$

From equation (6) an evolution equation for C^{mn} can be obtained and reads

$$\begin{aligned} \frac{\partial}{\partial t}C^{mn}(t,t') = &\, m\left\langle X^n(t')\left[x^{m-1}D_1(x,t,P)\right]_{x=X(t)}\right\rangle\\ &+m(m-1)\left\langle X^n(t')\left[x^{m-2}D_2(x,t,P)\right]_{x=X(t)}\right\rangle. \end{aligned} \quad (7)$$

[a] e-mail: tdfrank@uni-muenster.de

Likewise, we find that in the stationary case the correlation function $C^{mn}(z) = \langle X^m(t+z)X^n(t)\rangle_{\rm st}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z}C^{mn}(z) = m\left\langle X^n(t)\left[x^{m-1}D_1(x,t,P_{\rm st})\right]_{x=X(t+z)}\right\rangle_{\rm st} + m(m-1)\left\langle X^n(t)\left[x^{m-2}D_2(x,t,P_{\rm st})\right]_{x=X(t+z)}\right\rangle_{\rm st}. \quad (8)$$

If the right hand sides of equations (7) and (8) depend only on autocorrelation functions of lower order, that is, on functions $C^{n'm'}$ with $n' \leq n$ and $m' \leq m$, then we deal with a closed set of evolution equations from which the autocorrelation functions C^{nm} can be computed. Let us illustrate this point by an example.

Plastino and Plastino proposed the nonlinear Fokker-Planck equation [19]

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}\frac{\mathrm{d}U_0(x)}{\mathrm{d}x}P + Q\frac{\partial^2}{\partial x^2}P^q\,. \quad (9)$$

Equation (9) can be written as a free energy Fokker-Planck equation [20] of the form $\partial P/\partial t = (\partial/\partial x)\{P(\partial/\partial x)\delta F/\delta P\}$, where F denotes the free energy measure $F = U - QS_q$ that involves the linear internal energy functional $U[P] = \langle U_0(X)\rangle$ and the one-parametric entropy $S_q[P] = (1-q)^{-1}\int[P^q - P]\mathrm{d}x$ proposed by Tsallis [21,22]. Equation (9) and modifications of it have extensively been studied, see [23–32] and references therein. The drift and diffusion coefficients of equation (9) read $D_1(x,t,P) = -\mathrm{d}U_0(x)/\mathrm{d}x$ and $D_2(x,t,P) = QP^{q-1}$. In what follows, we restrict ourselves to consider solutions $P(x,t;u)$ of equation (9) for which $D_2'(x,t,u) = D_2(x,t,P)$ can be regarded as the second Kramers-Moyal coefficient of Markov diffusion processes with transition probability densities described by

$$\frac{\partial}{\partial t}P(x,t|x',t';u) = \frac{\partial}{\partial x}\frac{\mathrm{d}U_0(x)}{\mathrm{d}x}P(x,t|x',t';u) + Q\frac{\partial^2}{\partial x^2}P^{q-1}(x,t;u)P(x,t|x',t';u)\,. \quad (10)$$

Stochastic processes described by equations (9) and (10) can alternatively be obtained from the Ito-Langevin

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -\left.\frac{\mathrm{d}U_0(x)}{\mathrm{d}x}\right|_{X(t)} + \sqrt{Q}\left.P^{(q-1)/2}(x,t;u)\right|_{X(t)}\Gamma(t), \quad (11)$$

see also [23]. In what follows, we will consider a parabolic potential $U_0(x) = \gamma x^2/2$. Then, for $q = 1$ equations (9) and (10) define an Ornstein-Uhlenbeck processes [18]. For $q \in (1/3,1)$ equation (10) has the stationary distribution [33]

$$P_{st}(x) = \frac{D_{\rm st}}{\left[1+\gamma(1-q)x^2/[2qQ\ D_{\rm st}^{q-1}]\right]^{1/(1-q)}} \quad (12)$$

that describes a power law

$$P_{st}(x) \propto |x|^{-\frac{2}{1-q}} \quad (13)$$

for $|x| \to \infty$. Here, $D_{\rm st}$ is given by $D_{\rm st} = [\gamma/(2qQz_q^2)]^{1/(1+q)}$, where z_q is defined by $z_q = \sqrt{\pi/(1-q)}\Gamma[(1+q)/[2(1-q)]]/\Gamma[1/(1-q)]$. The mean of the stationary solution vanishes, $\langle X\rangle_{\rm st} = 0$, and the variance $K_{\rm st}$ is given by $K_{\rm st} = [2qQz_q^{(1-q)}/\gamma]^{2/(1+q)}/(3q-1)$ and is finite for $q \in (1/3,1)$. Consequently, in the stationary case equation (10) becomes

$$\frac{\partial}{\partial t}P(x,t|x',t';P_{\rm st}) = \gamma\frac{\partial}{\partial x}xP(x,t|x',t';P_{\rm st}) + QD_{\rm st}^{q-1}\frac{\partial^2}{\partial x^2}\left[1+\frac{\gamma}{2qQ\ D_{\rm st}^{q-1}}(1-q)x^2\right]P(x,t|x',t';P_{\rm st}). \quad (14)$$

We read off from equation (14) that in the stationary case and for $q \in (1/3,1)$ the diffusion coefficient $D_2(x,t,P) = QP^{q-1}$ can indeed be regarded as the second Kramers-Moyal coefficient of a Markov diffusion process. From equation (14) one can derive evolution equations for the stationary autocorrelation functions that correspond to special cases of equation (8). For example, we find

$$\frac{\mathrm{d}}{\mathrm{d}z}C^{11}(z) = -\gamma C^{11}(z) \quad (15)$$

for $C^{11}(z) = \langle X(t+z)X(t)\rangle_{\rm st}$ and

$$\frac{\mathrm{d}}{\mathrm{d}z}C^{22}(z) = -\frac{\gamma(3q-1)}{q}\left[C^{22}(z) - \left\langle X^2\right\rangle_{\rm st}^2\right] \quad (16)$$

for $C^{22}(z) = \left\langle X^2(t+z)X^2(t)\right\rangle_{\rm st}$. Solving these equations for the respective initial conditions, we get

$$C^{11}(z) = \left\langle X^2\right\rangle_{\rm st}\exp\{-\gamma z\} \quad (17)$$

and

$$C^{22}(z) = \left\langle X^2\right\rangle_{\rm st}^2 + \left[\left\langle X^4\right\rangle_{\rm st} - \left\langle X^2\right\rangle_{\rm st}^2\right]\exp\left\{-\frac{\gamma(3q-1)}{q}z\right\} \quad (18)$$

with $\left\langle X^2\right\rangle_{\rm st} = K_{\rm st}$. The amplitude of the exponential function in equation (18) is semi-positive because we have $\left\langle X^4\right\rangle_{\rm st} - \left\langle X^2\right\rangle_{\rm st}^2 = \left\langle [X^2 - \left\langle X^2\right\rangle]^2\right\rangle_{\rm st} \geq 0$. Consequently, the autocorrelation function $C^{22}(z)$ decays monotonically from $C^{22}(0) = \left\langle X^4\right\rangle_{\rm st}$ to $\lim_{z\to\infty} C^{22}(z) = \left\langle X^2\right\rangle_{\rm st}^2 = K_{\rm st}^2$.

Next, let us compare the analytical results obtained from the Fokker-Planck description with numerical results obtained from simulations of the Ito-Langevin equation (11). In general, we solve the Ito-Langevin equation (4) for $D_1'(x,t,u) = D_1(x,t,P)$ and $D_2'(x,t,u) = D_2(x,t,P)$ by means of the Euler forward scheme [18]

$$X_{n+1}^l = X_n^l + \Delta t\, D_1'(X_n^l,t_n,u) + \sqrt{D_2'(X_n^l,t_n,u)}\sqrt{\Delta t}\, w_n^l\,. \quad (19)$$

Here, the variables $X^l(t)$ denote realization of $X(t)$, time is given by $t_n = n\Delta t + t_0$ with $n = 0,1,2,\ldots$ and w_n^l are the realizations of random numbers w_n satisfying $\langle w_n\rangle$ and $\langle w_i w_k\rangle = 2\delta_{ik}$ [18]. The coefficients $D_1'(X_n^l,t_n,u)$ and

$D_2'(X_n^l, t_n, u)$, in turn, are computed from the realizations X_n^l with $l = 1, \ldots, L$ by means of

$$D_1'(x, t_n, u) = D_1\left(x, t_n, \frac{1}{L}\sum_{l=1}^{L}\delta(x - X_n^l)\right),$$
$$D_2'(x, t_n, u) = D_2\left(x, t_n, \frac{1}{L}\sum_{l=1}^{L}\delta(x - X_n^l)\right) \quad (20)$$

because of $P(x, t; u) = \langle\delta(x - X(t))\rangle$. Finally, we exploit the relation

$$\delta(x - x') = \frac{1}{\sqrt{2\pi}\Delta x}\exp\left\{-\frac{1}{2}\left[\frac{x - x'}{\Delta x}\right]^2\right\}, \quad (21)$$

which holds in the limit $\Delta x \to 0$ [34]. Then, equation (20) reads

$$D_1'(x, t_n, u) =$$
$$D_1\left(x, t_n, \frac{1}{\sqrt{2\pi}\Delta x L}\sum_{l=1}^{L}\exp\left\{-\frac{1}{2}\left[\frac{x - X_n^l}{\Delta x}\right]^2\right\}\right),$$
$$D_2'(x, t_n, u) =$$
$$D_2\left(x, t_n, \frac{1}{\sqrt{2\pi}\Delta x L}\sum_{l=1}^{L}\exp\left\{-\frac{1}{2}\left[\frac{x - X_n^l}{\Delta x}\right]^2\right\}\right). \quad (22)$$

Our simulation scheme based on equations (19) and (22) becomes exact in the limit $\Delta t \to 0$, $L \to \infty$, and $\Delta x \to 0$ (where first the limit $\Delta x \to 0$, second the limit $L \to \infty$, and third the limit $\Delta t \to 0$ has to be carried out). Therefore, L should correspond to a large number, whereas Δx and Δt should correspond to small numbers. Figure 1 shows the stationary distribution of the Plastino-Plastino Fokker-Planck equation as obtained from the analytical result (12) and a simulation of the Langevin equation (11). Note that the tails of the power-law distribution (12) become straight lines in the log-log plot shown in Figure 1. Figures 2 and 3 show the correlations functions $C^{11}(z)$ and $C^{22}(z)$ as obtained from our analytical considerations on the nonlinear Fokker-Planck equation (9) and as obtained by simulations of the Langevin equation (11).

From equation (7) or alternatively from equation (10) we can also derive nonstationary autocorrelation functions of the Plastino-Plastino model. For example, substituting $D_1 = -\gamma x$ into equation (7) we find

$$\frac{\partial}{\partial t}C^{11}(t, t') = -\gamma C^{11}(t, t'), \quad (23)$$

which eventually gives us

$$C^{11}(t, t') = \left\langle X^2(t')\right\rangle \exp\{-\gamma(t - t')\}. \quad (24)$$

By means of equation (24), we can compute the autocorrelation function C^{11} for every pair (t, t') with $t \geq t'$ provided that the value of the second moment $\left\langle X^2\right\rangle$ at time t' is given. For example, equation (9) is solved by the transient solution [33]

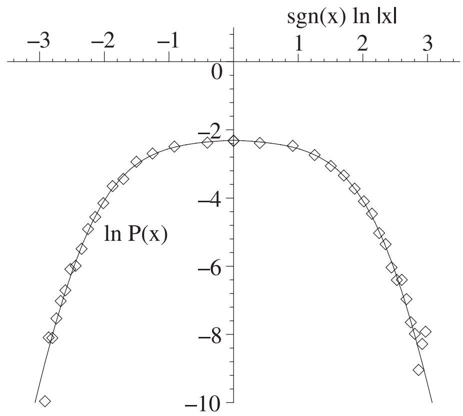

Fig. 1. Solid line: exact solution (12). Diamonds: P_{st} computed from the Langevin equation (11) for $U_0 = \gamma x^2/2$, $\gamma = 0.1$, $Q = 1.0$, $q = 0.8$.

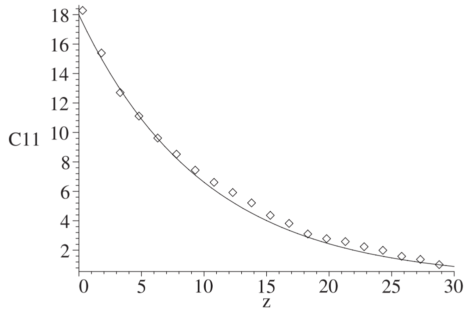

Fig. 2. Autocorrelation function C^{11} computed from equation (17) (solid line) and from the Langevin equation (11) with $U_0 = \gamma x^2/2$ (diamonds). Parameters as in Figure 1.

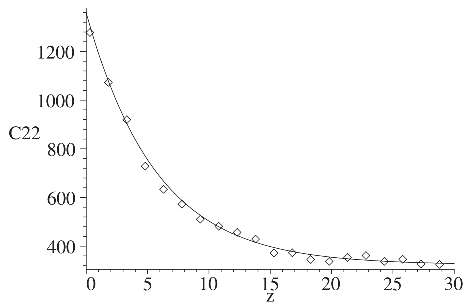

Fig. 3. Autocorrelation function C^{22} computed from equation (18) (solid line) and from the Langevin equation (11) with $U_0 = \gamma x^2/2$ (diamonds). Parameters as in Figure 1.

$$P(x, t; \delta(x - x_0)) =$$
$$\frac{D(t)}{[1+[z_q D(t)]^2(1 - q)[x - M_1(t)]^2]^{1/(1-q)}} \quad (25)$$

with

$$M_1(t) = x_0 \exp\{-\gamma(t - t_0)\}, \quad (26)$$

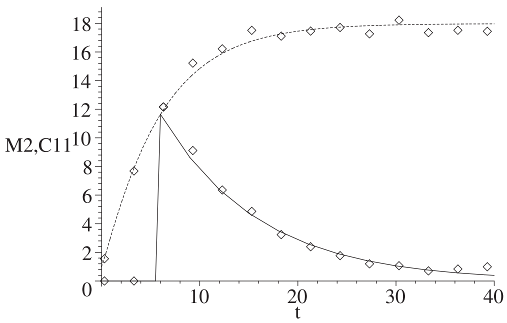

Fig. 4. Solid line: autocorrelation function $C^{11}(t,t')$ for $t'=6$ computed from equations (24, 26, 28). Dashed line: second moment $\langle X^2(t)\rangle$ computed from equations (26) and (28). Diamonds: $C^{11}(t,t')$ for $t'=6$ and $\langle X^2(t)\rangle$ obtained by solving numerically the Langevin equation (11) with $U_0=\gamma x^2/2$. For $t<t'$ we put $C^{11}=0$. Parameters: $x_0=-1$, $t_0=0$, and see Figure 1.

$$D(t) = \left[\frac{\gamma}{2qQ}\frac{1}{[z_q]^2}\;\frac{1}{1-\exp\{-(1+q)\gamma(t-t_0)\}}\right]^{1/(1+q)}. \quad (27)$$

and

$$K(t) = \frac{1}{3q-1}\left[\frac{2qQ[z_q]^{(1-q)}}{\gamma}\;(1-\exp\{-(1+q)\gamma(t-t_0)\})\right]^{2/(1+q)}. \quad (28)$$

Replacing t by t' in equations (26) and (28), we can compute $\langle X^2(t')\rangle = K(t') + M_1^2(t')$. Substituting this result into equation (24), we obtain an analytical expression for $C^{11}(t,t')$. Figure 4 shows the exact analytical results and the numerical results for $C^{11}(t,t')$ and $\langle X^2(t)\rangle$. The results are in excellent agreement.

We have derived analytical expressions for autocorrelation functions of nonlinear Fokker-Planck equations. In doing so, we have put the theory of nonlinear Fokker-Planck equations on a more equal footing with the theory of linear Fokker-Planck equations for which it is known that analytical expression for autocorrelation functions can be obtained. Moreover, the results derived here can be used for model selection. If we observe a particular (time-dependent) distribution function P, then there are usually several models at our disposal that can reproduce the distribution function. These models will most probably yield different correlation functions. As a result, by comparing not only the distribution functions but also the correlation functions of models with experimental data, we might be able to select a model out of a set of models that all do their job equally well as far as the reproduction of a distribution function is concerned. Since different models often involve different mechanisms, the study of correlations functions can be used to uncover the physical mechanisms involved in experimental observations.

Rapid Note

References

1. H. Daido, Phys. Rev. Lett. **87**, 048101 (2001)
2. W. Gerstner, J.L. van Hemmen, in *Models of neural networks II: Temporal aspects of coding and information processing in biological systems*, edited by E. Domany, J.L. van Hemmen, K. Schulten (Springer, Berlin, 1994), pp. 1–93
3. Y. Kuramoto, *Chemical oscillations, waves, and turbulence* (Springer, Berlin, 1984)
4. J.H. Li, P. Hänggi, Phys. Rev. E **64**, 011106 (2001)
5. S.H. Strogatz, Physica D **143**, 1 (2000)
6. P.A. Tass, *Phase resetting in medicine and biology - Stochastic modelling and data analysis* (Springer, Berlin, 1999)
7. T.D. Frank, A. Daffertshofer, C.E. Peper, P.J. Beek, H. Haken, Physica D **144**, 62 (2000)
8. T.D. Frank, A. Daffertshofer, P.J. Beek, Phys. Rev. E **63**, 011905 (2001)
9. T.D. Frank, A. Daffertshofer, P.J. Beek, J. Biol. Phys. **28**, 39 (2002)
10. C. van den Broeck, J.M.R. Parrondo, R. Toral, Phys. Rev. Lett. **73**, 3395 (1994)
11. K. Wiesenfeld, P. Colet, S.H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996)
12. A.A. Zaikin, J. Garcia-Ojalvo, L. Schimansky-Geier, J. Kurths, Phys. Rev. Lett. **88**, 010601 (2002)
13. T.D. Frank, Physica D **189**, 199 (2004)
14. H.C. Öttinger, *Stochastic processes in polymeric fluids* (Springer, Berlin, 1996)
15. M. Shiino, Phys. Rev. E **67**, 056118 (2003)
16. P.H. Chavanis, Phys. Rev. E **68**, 036108 (2003)
17. T.D. Frank, Physica A **331**, 391 (2004)
18. H. Risken, *The Fokker-Planck equation — Methods of solution and applications* (Springer, Berlin, 1989)
19. A.R. Plastino, A. Plastino, Physica A **222**, 347 (1995)
20. T.D. Frank, Phys. Lett. A **305**, 150 (2002)
21. C. Tsallis, J. Stat. Phys. **52**, 479 (1988)
22. C. Tsallis, Braz. J. Phys. **29**, 1 (1999)
23. L. Borland, Phys. Rev. E **57**, 6634 (1998)
24. L. Borland, F. Pennini, A.R. Plastino, A. Plastino, Eur. Phys. J. B **12**, 285 (1999)
25. L. Borland, Phys. Rev. Lett. **89**, 098701 (2002)
26. C. Giordano, A.R. Plastino, M. Casas, A. Plastino, Eur. Phys. J. B **22**, 361 (2001)
27. G. Kaniadakis, Physica A **296**, 405 (2001)
28. G. Kaniadakis, Phys. Lett. A **288**, 283 (2001)
29. E.K. Lenzi, R.S. Mendes, C. Tsallis, Phys. Rev. E **67**, 031104 (2003)
30. L.C. Malacarne, R.S. Mendes, I.T. Pedron, E.K. Lenzi, Phys. Rev. E **65**, 052101 (2002)
31. A. Rigo, A.R. Plastino, M. Carsas, A. Plastino, Phys. Lett. A **276**, 97 (2000)
32. C. Tsallis, D.J. Bukman, Phys. Rev. E **54**, R2197 (1996)
33. T.D. Frank, A. Daffertshofer, Physica A **285**, 351 (2000)
34. E.N. Moore, *Theoretical mechanics* (John Wiley and Sons, New York, 1983)